# Conserved Charge Corresponding to Lorentz Boost: Field Theorist vs. General Relativist

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We derive in details the conserved charges that correspond to Lorentz boosts, first via the well known method of field theorists, i.e., by the application of Noether's theorem, and secondly by considering the Killing vectors in Minkowski spacetime $\mathbb{R}^{3,1}$, a method favored by general relativists. Indeed, with both methods, we will derive the general form of the conserved charges correspond to Lorentz transformation, of which Lorentz boost is only a special case. Along the way, we review some important concepts and give some mathematical exercises for the ambitious readers.

## I. INTRODUCTION: GALILEAN BOOST

Before we go down the road of rigor, it is perhaps helpful to make a few observations and hand-waving arguments. First of all, let us take a step back from Special Relativity and consider the question:

> *What happens if we consider good old Newtonian physics, instead of Special Relativity? That is, what is the conserved charge corresponds to good old Galilean boost?*

In this case, the transformation from one inertial reference frame to another is related simply by $t \mapsto t$, $x \mapsto x + vt$ where $v$ is the relative constant velocity between the frames as usual. In momentum space then, momentum and position transformed by $p \mapsto p + mv$ and $q \mapsto q + vt$ respectively [For simplicity, consider first the moment $t = 0$], so that Galilean boost corresponds to a translation in phase space. This is actually quite natural if you recall from quantum mechanics that the momentum operator is $\hat{P} = -i\hbar\frac{d}{dx}$, i.e. momentum observable generates translation in position space. We all know that position and momentum are just Fourier transform pairs, so by symmetry, position observable likewise generates translation in momentum space. This is exactly what we saw earlier: Galilean boost corresponds to translation in momentum space, or put it slightly differently, the generator of Galilean boosts is the position observable $q$, or more precisely, $mq$. In many-particle case, we would expect that the generator is the total mass times the center of mass [All these can be made fully rigorous, see, e.g., [1]]. We would like to claim that this is a conserved quantity, much like momentum, as the generator of translation, is conserved.

However, if you pause and think for a while, you may object that this quantity is *not conserved* since the position of the center of mass is *time dependent* [we are boosting after all]! Previously we set $t = 0$, but we could have worked out the case for arbitrary $t$, in which case we would indeed get $mq - pt$. There is however no contradiction. Recall that we are looking at *boost* symmetry. This itself is *time-dependent*, so its "conserved quantity" turns out to also be time-dependent!

What we say about Galilean boost should be straightforwardly generalized to Lorentz boost, but let us do it rigorously.

## II. FIELD THEORIST: NOETHER'S THEOREM

For this section, we will follow the treatment in [2] closely. Recall that according to Noether's theorem, every continuous symmetry of the Lagrangian gives rise to a conserved current $j^\mu(x)$, $x \in \mathbb{R}^{3,1}$ such that the equations of motion implies $\partial_\mu j^\mu = 0$. Note that by symmetry we mean that after transformation, the Euler-Lagrange equation remains unchanged, this does *not* imply that the Lagrangian is unchanged.

Since we are dealing with continuous symmetry, we can work with infinitesimals: Consider a continuous infinitesimal transformation to a field $\phi(x)$:

$$\phi(x) \longmapsto \phi'(x) = \phi(x) + \epsilon\Delta\phi(x), \tag{1}$$

for some as yet unknown $\Delta\phi(x)$ and small $\epsilon$.

The Lagrangian transforms as

$$\mathcal{L}(x) \longmapsto \mathcal{L}'(x) + \epsilon\partial_\mu\mathcal{J}^\mu, \tag{2}$$

for some $\mathcal{J}^\mu$.

Since the Lagrangian depends on $\phi$ and $\partial_\mu\phi$, chain rule and product rule gives

$$\epsilon\Delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\epsilon\Delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\epsilon\Delta(\partial_\mu\phi) \tag{3}$$

$$= \frac{\partial\mathcal{L}}{\partial\phi}\epsilon\Delta\phi + \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\epsilon\Delta\phi\right] - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\epsilon\Delta\phi \tag{4}$$

$$= \epsilon\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi\right] + \epsilon\left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\right]\Delta\phi. \tag{5}$$

By Euler-Lagrange equation [which we assumed invariant under the infinitesimal transformation], only the first term remains. Therefore

$$\partial_\mu\mathcal{J}^\mu = \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi\right], \tag{6}$$

or equivalently,

$$\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi - \mathcal{J}^\mu\right] = 0. \tag{7}$$

The conserved current is then defined by

$$\boxed{j^\mu := \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi - \mathcal{J}^\mu.} \tag{8}$$

We further define the [conserved] *charge* by

$$\boxed{Q := \int d^3x\ j^0.} \tag{9}$$

This is conserved since

$$\partial_\mu j^\mu = 0 \Rightarrow \partial_t j^t - \partial_i j^i = 0 \tag{10}$$
$$\implies \frac{d}{dt}\int d^3x j^0 - \int d^3x \partial_i j^i = 0 \tag{11}$$
$$\implies \frac{d}{dt}\int d^3x j^0 = \frac{dQ}{dt} = 0. \tag{12}$$

The last implication follows from the implicit assumption that $j^\mu$ is compactly supported [physically, this assumes the current decays sufficiently fast towards infinity], so that total derivative term vanishes under the integral.

Having revised the standard stuffs above, we now consider infinitesimal form of the Lorentz transformations

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu, \tag{13}$$

where $\omega^\mu{}_\nu$ is infinitesimal. By the definition of Lorentz transformation $\Lambda^T\eta\Lambda = \eta$ where $\eta$ is the standard Minkowski metric, we have

$$(\delta^\mu{}_\alpha + \omega^\mu{}_\alpha)(\delta^\nu{}_\beta + \omega^\nu{}_\beta)\eta^{\alpha\beta} = \eta^{\mu\nu}. \tag{14}$$

Expanding out and keeping only first power in $\omega^\mu{}_\nu$, we can obtain the condition

$$\omega^{\mu\nu} + \omega^{\nu\mu} = 0. \tag{15}$$

That is, the infinitesimal form $\omega^{\mu\nu}$ of the Lorentz transformation is necessarily anti-symmetric.

Consider now a scalar field $\phi(x)$. Under Lorentz transformation, its transformation law [$\phi(x) = \phi'(x')$] satisfies

$$\phi(x) \mapsto \phi'(x) = \phi(\Lambda^{-1}x) \tag{16}$$
$$= \phi(x^\mu - \omega^\mu{}_\nu x^\nu) \tag{17}$$
$$= \phi(x^\mu) - \omega^\mu{}_\nu x^\nu \partial_\mu\phi(x), \tag{18}$$

where the last implication follows from neglecting higher order terms in the power series of $\phi$. This means that $\epsilon\delta\phi = -\omega^\mu{}_\nu x^\nu\partial_\mu\phi(x)$. Since the Lagrangian [density] is a scalar quantity, we likewise have

$$\epsilon\Delta\mathcal{L} = -\omega^\mu{}_\nu x^\nu\partial_\mu\mathcal{L} = -\partial_\mu(\omega^\mu{}_\nu x^\nu\mathcal{L}). \tag{19}$$

The last equality is due to the fact that $\omega^{\mu\nu}$ is anti-symmetric.

We can now find the conserved current via Eq.(8):

$$\epsilon j^\mu = -\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\omega^\alpha{}_\nu x^\nu\partial_\alpha\phi + \omega^\mu{}_\nu x^\nu\mathcal{L} \tag{20}$$
$$= -\omega^\alpha{}_\nu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}x^\nu\partial_\alpha\phi - \delta^\mu{}_\alpha x^\nu\mathcal{L}\right] \tag{21}$$
$$= -\omega^\alpha{}_\nu T^\mu{}_\alpha x^\nu, \tag{22}$$

where

$$\boxed{T^\mu_{\ \nu} := \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi - \delta^\mu{}_\nu\mathcal{L}} \tag{23}$$

defines the *energy-momentum tensor*, such that

$$\boxed{E := \int d^3x\ T^{00},\quad p^i := \int d^3x\ T^{0i}} \tag{24}$$

are the total energy and total momentum of the field configuration respectively. Removing the infinitesimal factor [hence freeing up two more indices], the conserved current is really

$$(j^\mu)^{\alpha\beta} = x^\alpha T^{\mu\beta} - x^\beta T^{\mu\alpha}. \tag{25}$$

This satisfies $\partial_\mu(j^\mu)^{\alpha\beta} = 0$, and so there are $3\times 2 = 6$ conserved currents, and hence also 6 conserved charges.

For $\alpha,\ \beta \in \{1,2,3\}$, the Lorentz transformation is a spatial rotation and the three conserved charges give the total angular momentum of the field:

$$Q^{ij} = \int d^3x\ \left(x^i T^{0j} - x^j T^{0i}\right) = x^i p^j - x^j p^i. \tag{26}$$

This is nothing but our familiar expression of *angular momentum*. What about the boost part? Well, it is simply

$$Q^{0i} = \int d^3x\ \left(x^0 T^{0i} - x^j T^{0i}\right). \tag{27}$$

Since this is conserved, we have

$$0 = \frac{dQ^{0i}}{dt} \tag{28}$$
$$= \int d^3x\ T^{0i} + t\int d^3x\ \frac{\partial T^{0i}}{\partial t} - \frac{d}{dt}\int d^3x\ x^i T^{00} \tag{29}$$
$$= p^i + t\frac{dp^i}{dt} - \frac{d}{dt}\int d^3x\ x^i T^{00}. \tag{30}$$

The second term vanishes since momentum $p^i$ is conserved. Thus we conclude that

$$\frac{d}{dt}\int d^3x\ x^i T^{00} = \text{const.} \tag{31}$$

That is, the center of energy of the field travels with constant velocity. We can find the exact expression for the conserved charge $Q^{0i}$. This is given by, in the unit $c = 1$,

$$Q^{0i} = x^0 p^i - x^i E = (t\vec{p} - \vec{x}E)^i = \gamma m\left(\vec{v}t - \vec{x}\right)^i. \tag{32}$$

That is, the conserved charge is the center of mass multiplied by mass times the Lorentz factor $\gamma$, just like what we would expect from our discussion on the conserved charged corresponding to Galilean boost.

## III. GENERAL RELATIVIST: KILLING VECTORS

In general relativity, symmetry means isometry. Given a timelike Killing vector, there corresponds a conservation law of some kind. Note that given a spacetime manifold, the existence of a Killing vector is *not* automatic. Therefore, symmetry is generically a rare thing in general relativity. For example, energy conservation does *not* hold. Back in good old quantum field theory on flat spacetime, when we say that energy is conserved, this is traced back to the fact that there exists time translation symmetry. In an expanding universe for example, there is no such thing as time translation symmetry, and hence also no energy conservation.

We first recall the meaning of Killing vectors. Let $M$ be a semi-Riemannian manifold, i.e. the tangent space $T_pM$ at any point in $M$ is isometric to $\mathbb{R}^{3,1}$. We call $M$ a *spacetime*. Let $X, Y$ be vector fields on $M$. Recall that the *Lie derivative* $\pounds_X Y$ is another vector field [and thus is a derivation that acts on smooth functions] such that

$$(\pounds_X Y)f = [X, Y]f = (XY - YX)f; \quad \forall f \in C^\infty(M). \tag{33}$$

That is, if you are not familiar with Lie derivative, just think of it as a Lie bracket defined on the tangent space. Now, just like the more familiar covariant derivative [to be reviewed below], the Lie derivative can be extended to tensors of any type by requiring that $\pounds_X f = Xf$ and compatibility with contractions. E.g. let $Y$ be a $(1,0)$-tensor [i.e. vector] and $\omega$ be a $(0,1)$-tensor [i.e. one-form], then $\omega(Y)$ is a $(0,0)$-tensor, i.e. a scalar. If we demand Leibnizian property to hold, then

$$0 = \pounds_X(\omega(Y)) = X(\omega(Y)) = (\pounds_X \omega)Y + \omega(\pounds_X Y). \tag{34}$$

This means that we should define the Lie derivative of one-form by:

$$(\pounds_X \omega)(Y) := X(\omega(Y)) - \omega(\pounds_X Y). \tag{35}$$

**Definition:** A vector field $X$ is called a *Killing vector field* with respect to the metric $g$ if $\pounds_X g = 0$.

Essentially, this is saying that geometry determined by the metric does not change if we move in the flow of the Killing vector field. More technically, we note that if $G_t : (M, g) \longrightarrow (M, g)$ is a family a diffeomorphism determined by the vector field $X$, and $G_t^* g = g$, then $\frac{d}{dt}|_{t=0} = \pounds_X g$.

We quickly recall properties of *covariant derivatives*: Given any vector fields $X$ abd $Y$, the covariant derivative $\nabla_X Y(p)$ depends only on $X(p)$. We have

$$\nabla_{f_1 X_1 + f_2 X_2} Y = f_1 \nabla_{X_1} Y + f_2 \nabla_{X_2} Y, \tag{36}$$

and

$$\nabla_X(fY) = (Xf)Y + f\nabla_X Y. \tag{37}$$

Like the Lie derivative, $\nabla_X$ can be extended to tensors of any type, by requiring that $\nabla_X f = Xf$ and compatibility with contractions.

**Exercise:** Compare and contrast various notions of derivatives in differential geometry: Lie derivative, covariant derivative, exterior derivative. Make sure you understand them! While you are at it, also look at the idea of *anti-derivation*, and in particular, the notion of interior product. While physicists don't usually pay extra care on the spaces these operators are defined, you should try to do so! For example, what should take the places of heart and spade in $\nabla_X : \heartsuit \longrightarrow \spadesuit$?

**Remark:** Note that $\nabla_X(fY) \neq f\nabla_X(Y)$, so $\nabla_X(W)$ is *not* a tensor in $W$! If you are introduced to tensors by its transformation laws without knowing what tensors *are*, you will be confused! But if you know that tensors are multilinear map, this would be obvious! However, of course any covariant derivative of a tensor field is a tensor!

**Lemma:** $(\pounds_X g)_{\mu\nu} = \nabla_\nu X_\mu + \nabla_\mu X_\nu$.
*Proof:*

$$(\pounds_X g)_{\mu\nu} = X(g_{\mu\nu}) - g\left(\pounds_X \frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right) - g\left(\frac{\partial}{\partial x^\mu}, \pounds_X \frac{\partial}{\partial x^\nu}\right) \tag{38}$$

$$= \left[\underbrace{(\nabla_X g)_{\mu\nu}}_{=0} + g\left(\nabla_X \frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right) + g\left(\frac{\partial}{\partial x^\mu}, \nabla_X \frac{\partial}{\partial x^\nu}\right)\right] - g\left(\left[X, \frac{\partial}{\partial x^\mu}\right], \frac{\partial}{\partial x^\nu}\right) - g\left(\frac{\partial}{\partial x^\mu}, \left[X, \frac{\partial}{\partial x^\nu}\right]\right) \tag{39}$$

$$= g\left(\nabla_X \frac{\partial}{\partial x^\mu} - \left[X, \frac{\partial}{\partial x^\mu}\right], \frac{\partial}{\partial x^\nu}\right) - g\left(\frac{\partial}{\partial x^\mu}, \nabla_X \frac{\partial}{\partial x^\nu} - \left[X, \frac{\partial}{\partial x^\nu}\right]\right) \tag{40}$$

$$= g\left(\nabla_{\partial_\mu} X, \frac{\partial}{\partial x^\nu}\right) + g\left(\frac{\partial}{\partial x^\mu}, \nabla_{\partial_\nu} X\right) \quad \text{since torsion free condition implies } \nabla_X Y - \nabla_Y X = [X, Y], \tag{41}$$

$$= g\left(X^\rho_{;\mu} \frac{\partial}{\partial x^\rho}, \frac{\partial}{\partial x^\nu}\right) + g\left(\frac{\partial}{\partial x^\mu}, X^\rho_{;\nu} \frac{\partial}{\partial x^\rho}\right) \tag{42}$$

$$= \nabla_\nu X_\mu + \nabla_\mu X_\nu. \tag{43}$$

We thus have the following result:

$$\boxed{\text{Killing Equation: } \nabla_\nu X_\mu + \nabla_\mu X_\nu = 0.} \tag{44}$$

We then have the following theorem:
**Theorem:** The space of Killing fields of $(\mathbb{R}^{3,1}, \eta)$ is a 10-dimensional vector space.

For a proof, consider canonical [global] coordinates $(t, x, y, z)$ on $\mathbb{R}^{3,1}$. For Minkowski space, covariant derivative reduces to the usual partial derivatives and so the Killing equation is $\partial_\mu X_\nu + \partial_\nu X_\mu = 0$. Obviously, for any fixed $\nu$, the constant vector fields $X^\mu_{(\nu)} = \delta^\mu_\nu$, $i = 0, 1, 2, 3$ are trivial solutions to the Killing equation. These Killing vectors give rise to spacetime translations along their respective coordinate axes.

Consider now the ansatz $X_\nu = C_{\mu\nu} x^\mu$, some linear combinations of coordinate vector fields, where $C_{\mu\nu} = C_{\mu\nu}(x^\mu, x^\nu)$. Then, we have, by the Killing equation,

$$\partial_\mu (C_{\alpha\nu} x^\alpha) + \partial_\nu (C_{\alpha\mu} x^\alpha) = 0 \tag{45}$$

$$\Rightarrow C_{\alpha\nu} \partial_\mu x^\alpha + C_{\alpha\mu} \partial_\nu x^\alpha = 0 \tag{46}$$

$$\Rightarrow C_{\mu\nu} = -C_{\nu\mu}. \tag{47}$$

That is, $C_{\mu\nu}$ is antisymmetric in $\mu \longleftrightarrow \nu$. Since ${}^4C_2 = 6$, there are 6 independent solutions of this form, all of which are linearly independent of the trivial Killing vectors $X^\mu_{(\nu)} = \delta^\mu_\nu$. Thus, the dimension of this vector space is *at least* 10.

**Exercise:** Complete the proof that the dimension of vector space of linearly independent Killing vectors is indeed 10 in Minkowski space. Since we have shown above that the dimension is *at least* 10, you can finish the proof by showing that the dimension is *at most* 10. This can be done by proving the general result that the Lie algebra of Killing vector fields on a connected semi-Riemannian manifold of dimension $n$ is at most $n(n+1)/2$. Hint: Fix $p \in M$. Let $E$ be the map that sends each Killing vector field $X$ to $(X_p, (\nabla X)_p)$ where $(\nabla X)(Y) = \nabla_X Y$ for all smooth vector fields $X$ and $Y$. That is, $E$ is a linear transformation from the Lie algebra of Killing vector fields to $T_pM \times \mathfrak{o}(T_pM)$, where $\mathfrak{o}(T_pM)$ is the Lie algebra consisting of all skew-adjoint linear operators on $T_pM$. Use elementary facts about linear transformations to argue that the dimension must be at most $n(n+1)/2$. Remark: If the upper bound is saturated, such spacetime is said to be *maximally symmetric*.

One can easily verify explicitly, that among the 6 independent solutions, there are 3 with components of the form

$$Y^l_{(k)} = \sum_m \epsilon^{lkm} x^m, \quad k, l \in \{1, 2, 3\}, \quad Y^0_{(k)} = 0, \tag{48}$$

which corresponds to spatial rotations about the $x^k$-axis. For example,

$$Y_3 = \underbrace{Y_3^2}_{=x^1} \frac{\partial}{\partial x^2} - \underbrace{Y_3^1}_{=x^2} \frac{\partial}{\partial x^1}. \tag{49}$$

There are 3 other Killing vectors with components the forms

$$X^i_{(k)} = \delta^i_k t; \quad X^0_{(k)} = x^k. \tag{50}$$

This corresponds to the Lorentz boosts. More explicitly, the Lorentz boosts correspond to the Killing vectors

$$t \frac{\partial}{\partial x^k} + x^k \frac{\partial}{\partial t}. \tag{51}$$

For example, boosting along $x^1$-axis corresponds to

$$t \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial t} = \delta^1_1 t \frac{\partial}{\partial x^1} + X^0_{(1)} \frac{\partial}{\partial t}. \tag{52}$$

One should compare Eq.(51) with Eq.(32) and see that we are almost done with getting the conserved charge for Lorentz boost. If we assume $(+,-,-,-)$ signature, we can consider first the following result:

**Corollary:** Let $K$ be a Killing vector field, and $x(\tau)$ be a geodesic. Then the quantity $K_\mu \dot{x}^\mu$ is constant along the geodesic. The proof is straightforward:

$$\frac{d}{d\tau}(K_\mu \dot{x}^\mu) = (\nabla_\tau K_\mu)\dot{x}^\mu + K_\mu \nabla_\tau \dot{x}^\mu \tag{53}$$

$$= \nabla_\nu K_\mu \dot{x}^\nu \dot{x}^\mu + 0 \tag{54}$$

$$= \frac{1}{2}(\nabla_\nu K_\mu + \nabla_\mu K_\nu)\dot{x}^\mu \dot{x}^\nu = 0. \tag{55}$$

Thus, we take Eq.(51), and take inner product with the geodesics $x^\mu(\tau)$, with $\dot{x}(\tau) \equiv u(\tau)$. Then we obtain $g\left(t\frac{\partial}{\partial x^k} + x^k \frac{\partial}{\partial t}, u\right) = g_{ij} t \delta^i_k u^j + g_{00} x^k \delta^0_0 u^0 = -tp^k + x^k E$, where $u^k = p^k/m$, $u^0 = E/m$. Thus the conserved charge agrees with that of Eq.(32).

## IV. AN ALMOST TRIVIAL OBSERVATION

An alternative, non-rigorous way to find the conserved charge corresponding to Lorentz boost is as follows: Recall that for rotation in $\mathbb{R}^3$, the conserved charges are angular momenta. WLOG, say if we consider rotation around the $z$-axis, then the angular momentum $xp_y - yp_x$ is conserved. Now recall that in 4-dimensional spacetime rotation is described by matrix of the following form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{56}$$

On the other hand, we know that Lorentz boost in the $x$-direction, say, is given by the matrix

$$\begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{57}$$

where $\tanh\phi = v$. Of course, $v$ is really $v/c$, with $c = 1$ in our unit.

**Exercise:** This question is not usually raised in typical relativity course: why do we bother with introducing $\phi$ as a variable in place of $v$? The reason is that velocity addition is not linear in Special Relativity and this complicates calculation. Explicitly [for simplicity, consider 2-dimensional Lorentz boost], the Lorentz boost is given by the matrix

$$\Lambda(v) = \begin{pmatrix} \gamma & -v\gamma \\ -v\gamma & \gamma \end{pmatrix}. \tag{58}$$

Verify that this satisfies

$$\Lambda(v_1)\Lambda(v_2) = \Lambda\left(\frac{v_1 + v_2}{1 + v_1 v_2}\right). \tag{59}$$

We want to find a variable that allow us to add linearly, call it $\phi$. Particle physicists love to call this "rapidity". Mathematically, we want to find a function $v = f(\phi)$ which is one-to-one and satisfies the *homomorphism*

$$\Lambda(f(\phi_1))\Lambda(f(\phi_2)) = \Lambda(f(\phi_1) + f(\phi_2)). \tag{60}$$

That is,

$$f(\phi_1 + \phi_2) = \frac{f(\phi_1) + f(\phi_2)}{1 + f(\phi_1) f(\phi_2)}. \tag{61}$$

This suggests the choice $f(\phi) = \tanh\phi$. If we require that $v$ and $\phi$ agrees at low velocity, prove that this choice is in fact *unique*. Hint: Show that $f'(\phi) = 1 - f^2(\phi)$, where prime denotes derivative with respect to $\phi$. You can then simply appeal to the uniqueness theorem of ODE.

Thus we see that in place of spatial rotation, we now have hyperbolic rotation in $\mathbb{R}^{3,1}$. Instead of preserving circle, this rotation preserves hyperbola. Thus, instead of angular momentum

$$xp_y - yp_x, \tag{62}$$

we now have, analogously, simply

$$tp_x - xp_t = tp_x - xE. \tag{63}$$

Remember that energy is essentially "momentum in the time direction"!


[1] D.E. Soper, Galilean Boost Symmetry.
[2] D. Tong, *Quantum Field Theory: University of Cambridge Part III Mathematical Tripos*, Chapter 1.